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MASS TRANSFER THROUGH LAMINAR BOUNDARY LAYERS—1. THE VELOCITY BOUNDARY LAYER

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Abstract—The paper presents a method for calculating the distributions of laminar boundary-layer thickness and wall shear stress on a two-dimensional or axi-symmetrical body when the distributions of free-stream velocity and of mass transfer through the wall are specified. The method is a development of those of Walz (1941) and Thwaites (1949), and makes use of the "similar" solutions of the boundary-layer equations. Examples of the use of the method are given. The ordinary differential equation of the "similar" boundary layers is developed in a novel way.

Résumé—Cet article présente une méthode de calcul des distributions des épaisseurs de la couche limite laminaire et du frottement à la paroi pour un corps bi-dimensionnel ou de révolution, quand on connait les distributions de vitesses au loin et le transport de masse à travers la paroi. La méthode est un développement de celles de Walz (1941) et Thwaites (1949) et fait usage des solutions "similaires" des équations de la couche limite. Des exemples d'utilisation de cette méthode sont donnés. L'équation différentielle ordinaire des couches limites "semblables" est développée d'une façon nouvelle.

Zusammenfassung—Es wird eine Methode zur Berechnung der Verteilungen der laminaren Grenzschichtdicken und der Wandschubspannungen an einem zweidimensionalen oder achsialsymmetrischen Körper mitgeteilt, wenn die Verteilungen der Freistromgeschwindigkeit und der Stoffübertragung durch die Wand vorgeschrieben sind. Die Methode ist eine Weiterentwicklung jener von Walz (1941) und Thwaites (1949) und macht von den „ähnlichen" Lösungen der Grenzschichtgleichungen Gebrauch. Anwendungsbeispiele werden mitgeteilt. Die gewöhnliche Differentialgleichung der „ähnlichen" Grenzschichten ist auf einem neuen Weg abgeleitet.

Аннотация—В статье излагается метод расчета толщины ламинарного пограничного слоя, напряжения, трения и перенос массы для поверхности двухразмерного или осесимметричного тела при заданных скоростях свободного потока. Настоящий метод является развитием метода Уолза (1941) и Фуейтса (1949) и позволяет использовать «подобные» решения уравнений пограничного слоя. Приводятся примеры использования этого метода. Дан новый вывод обычного дифференциального уравнения «подобных» пограничных слоёв.

NOTATION

a, A constant (see equation (20)) (—);

C, A constant (see equation (4)) (various);

d, A constant (see equation (20)) (—);

E_2, "Correction function" (see equation (20)) (—);

Eu, Euler number (see Section 4.1 and equation (72)) (—);

f, Dimensionless stream function (see equation (58) (—);

$_0$, Dimensionless measure of the mass transfer rate (see equation (62)) (—);

F_2, Function representing rate of growth of momentum thickness δ_2 (see equation (10)) (—);

g_0, Constant in Newton's Second Law (lb_m ft/lb_f h^2);

H_{12}, Boundary-layer thickness ratio, "shape factor" (see equation (8)) (—);

H_{24}, Boundary-layer thickness ratio (see equation (9)) (—);

K, A constant (see Section 2.4, Case (c)) (various);

L, Reference length of surface (ft);

$\dot{m}''$, Mass transfer rate per unit area (lb_m/ft^2h);
n, A constant (see equation (4)) (—);
R, Distance of point on surface from axis of symmetry (ft);
R_0, A reference radius (ft);
u, Flow velocity in x-direction (ft/h);
U, Reference flow velocity (ft/h);
v, Flow velocity in y-direction (ft/h);
x, Distance measured along wall in same direction as mainstream (ft);
y, Distance measured normal to the wall (ft);
β, Parameter relating to acceleration of mainstream (see equation (57)) (—);
δ_1, Displacement thickness (see equation (5)) (ft);
δ_2, Momentum thickness (see equation (6)) (ft);
δ_4, Shear thickness (see equation (7)) (ft);
ζ, Dimensionless stream function (see equation (54)) (—);
η, Dimensionless space co-ordinate (see equation (59)) (—);
ν, Kinematic viscosity of fluid (ft^2/h);
ξ, Dimensionless space co-ordinate (see equation (55)) (—);
ρ, Fluid density (lb_m/ft^3);
τ, Shear stress in fluid (see equation (18)) (lb_f/ft^2);
ψ, Stream function (see equation (49)) (ft^2/h);
μ, Dynamic viscosity of fluid (see equation (18)) ($lb_m/ft\ h$).

Subscripts
G, Stream conditions just outside boundary layer;
S, Fluid conditions adjacent the surface.

Superscripts
*, Dimensionless form for two-dimensional boundary layers (see equations (13) to (16));
**, Dimensionless form for axi-symmetrical boundary layers (see equations (25) to (28));
—, Equivalent values in two dimensions according to Mangler's transformation (see equations (75) to (78)).

1. INTRODUCTION

1.1. *Outline of the general laminar mass transfer problem*

THE prediction of a mass transfer rate from first principles involves the solution of a partial differential equation expressing the conservation and flux of some appropriate fluid property [11]. This equation contains the fluid velocity: it can therefore only be solved if the velocity distribution is known.

The fluid velocity distribution can be predicted, in general, by solving a partial differential equation expressing the principles of conservation of matter and momentum. This solution must therefore be obtained prior to, or simultaneously with, that of the first-mentioned equation.

It is rare for either of the equations to be solved exactly in situations of practical interest, for they are both parabolic, necessitating numerical step-by-step solution. However, rather satisfactory approximate methods of solution are available: these rest on simplifying assumptions which reduce each parabolic equation to a first-order ordinary equation.

The present paper is the first of a series under preparation by the author and his co-worker, H. L. Evans. The purpose of the series is to provide the procedures, graphs and tables which are needed for the calculation of the mass transfer conductance appropriate to a laminar boundary layer; we shall thus be considering the fluid-dynamic half of the standard mass transfer problem. The other half, namely the calculation of the magnitude of the driving force appropriate to the particular thermodynamic properties of the materials, will not be our concern here: the procedures to be derived can thus be regarded as equally valid for vaporization, absorption, condensation, fuel-combustion, transpiration-cooling, or any other mass transfer process. In the first instance, only steady forced-convection boundary layers with uniform fluid properties will be considered.

1.2. *Outline of the present paper*

The above discussion has emphasized that a mass transfer prediction necessitates, in general, the obtaining of a solution of the differential equation governing velocity. This necessity remains, in some cases, when the approximate methods are used.† The first two papers in the series are therefore devoted to the velocity equation. In the present paper, the method of solution is described and explained; the following paper [15] contains the auxiliary functions and some discussion of their derivation.

The problem is conveniently formulated in terms of the boundary-layer thickness; it then runs: "Given the shape of the surface, the distribution of the mass transfer rate through that surface, and the distribution of the main-stream velocity over the surface; find the boundary-layer thickness at any point on the surface".

The method of solution which will be presented contains the following elements: (i) A simplifying assumption which leads to an ordinary differential equation connecting the boundary-layer thickness with quantities specified in the data. (ii) The use of vectorial dimensional analysis to give shape to the groups arising in the differential equation. (iii) Reference to the family of "similar" boundary layers for the quantitative links (functions) between these groups. (iv) Simplified procedures for the numerical solution of the differential equation.

The method is described in Section 2, and illustrated by examples in Section 3. The paper concludes with a discussion of the "similar" solutions of the partial differential equations of the boundary layer, from which the auxiliary functions are derived (Section 4). The latter treatment differs somewhat from that of Mangler [5] and Schlichting [9], being free of some of the difficulties inherent in the conventional treatment.

Relation to earlier work. The recommended method of solution should be regarded as a development of the methods of Walz [14] and of Thwaites [12]. These authors implicitly used the assumption referred to under (i) above, derived the requisite functions from the "similar" solutions as referred to under (iii) above, and used the linearizing approximation which is the starting-point for some of the procedures of (iv).

† In the terminology to be used later in the series, solutions to the velocity equation are needed for all Class II methods, and for some (not the best) Class I methods, of calculating heat and mass transfer.

The main new contribution of the present paper lies in its coverage of a much wider range of conditions by reason of the introduction of the influence of mass transfer; Walz and Thwaites could only use a single curve of the family shown in Fig. 2 for example, namely that corresponding to $v_S\delta_2/\nu = 0$. However the opportunity has also been taken to streamline the argument, for example in step (ii), to improve the accuracy of the computational procedure, and to provide some examples.

It may also be worthwhile pointing out the connexion between the present method and that of von Kármán [4] and Pohlhausen [7], as generalized for treating boundary layers with blowing and suction by Schlichting [8], Torda [13] and others.

The connexion may be expressed as follows: The Kármán–Pohlhausen method is similar in spirit to the present method and is indeed its progenitor. However the Kármán–Pohlhausen method was developed at a time when very few of the similar solutions of the boundary-layer equations were available: approximations to the required auxiliary functions therefore had to be obtained by the use of a "profile" method (assuming that the velocity profile could be represented by a polynomial, obtaining relations between the coefficients by reference to selected boundary conditions, and solving the differential equation in integrated form).

The Kármán–Pohlhausen method therefore contains two sources of error: (i) that associated with the assumption that the boundary layer "forgets" all about its origin and responds only to local influences (Section 2.2); (ii) that associated with the use of approximate auxiliary functions.

The procedure recommended in the present paper on the other hand, exploiting as it does the progress in boundary-layer theory which has been made in the last thirty years, contains only the first of the two sources of error.

2. THE METHOD OF CALCULATION

2.1. *The mathematical problem*

Figure 1 shows a two-dimensional body immersed in a plane steady laminar fluid stream. At a distance x along the surface from the stagnation point, the velocity normal to the surface is v_S, the stream velocity outside the boundary layer is u_G and the boundary-layer thickness is δ.

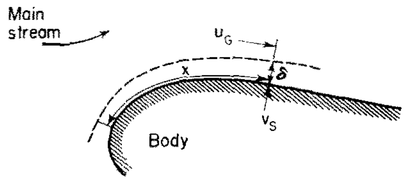

FIG. 1. Laminar boundary layer on a two-dimensional body.

Later, it will be necessary to give δ a precise definition; for the moment however we shall regard it as *a* measure of the boundary-layer thickness without specifying which one. This is done because the argument to be used applies to any definition.

The wall velocity v_S is related to the mass transfer rate of the standard formulation, $\dot{m}''$ [11], by the equation:

$$v_S = \dot{m}''/\rho_S \quad (1)$$

where $\dot{m}''$ = total mass flux through the interface (lb_m/ft^2h),
ρ_S = density of fluid adjacent the interface (lb_m/ft^3).

Since we shall assume the density to be uniform throughout the fluid, it is more convenient to use velocities than mass fluxes.

In the problem to be considered, v_S, like u_G, will be regarded as a known function of x. This may seem surprising in view of the fact that the end-objective is the calculation of the mass transfer rate, i.e. of v_S. The explanation is that, just as knowledge of the velocity distribution is required if the equation leading to the mass transfer rate is to be solved (mentioned in Section 1.1), so the latter quantity influences the former. The two problems must therefore be solved simultaneously; usually a convergent iterative technique is used.

The present mathematical problem may now be stated symbolically as:

Given: $u_G(x)$, $v_S(x)$, ν

Find: $\delta(x)$

where ν is the kinematic viscosity (ft^2/h).

Solution of this problem will be shown to provide a foundation for the subsequent calculations of the distribution of the mass transfer rate.

It should be noted that the discussion of a plane-flow problem simultaneously serves for problems in which the body and stream are axi-symmetrical; this will be explained in Section 2.5.

2.2. *The differential equation of boundary-layer growth*

Familarity with the experimental facts concerning boundary layers focusses attention on the fact that the boundary-layer thickness at any point on the surface is determined by the cumulative effect of upstream events, but is uninfluenced by what happens downstream. Moreover, experience with the solutions to partial parabolic equations, such as that for unsteady heat conduction, teaches that irregularities in profile (e.g. temperature distribution) are soon "ironed out"; the solutions may be said to have a poor memory for detail.

Now the partial differential equation of motion in the boundary layer is parabolic. We may therefore embody the experiences just referred to in the following assumption; this is the starting-point of the present method.

The basic simplifying assumption. We shall assume that the local rate of growth of the boundary-layer thickness δ, i.e. $d\delta/dx$, is determined solely by quantities which may be measured at the particular location on the surface. We shall include in the quantities for consideration all those mentioned so far, namely u_G, v_S and δ (but not x, since this cannot be measured without reference to other locations, viz. the stagnation point), and also the local gradient of the stream velocity du_G/dx. These choices imply that the boundary-layer thickness at any point is dependent in part on the thickness of the boundary layer immediately upstream but on no other upstream boundary-layer property. In other words, δ is regarded as being the only local property large enough to be "remembered" for an appreciable distance downstream; details of the shape of the velocity profile are assumed to be "forgotten" immediately.

The assumptions can be expressed mathematically in the following form:

$$\frac{d\delta}{dx} = f\left(u_G, v_S, \delta, \frac{du_G}{dx}, \nu\right) \tag{2}$$

where $f(\ldots)$, here and throughout the paper, signifies "some function" of the quantities in the bracket.

Dimensional analysis. The variables appearing in equation (2) must be grouped together so that the equation is dimensionally homogeneous. This can be carried out by the procedures of dimensional analysis. The maximum simplification of equation (2) is obtained if we incorporate the essential assumption of boundary-layer theory, namely that viscosity is only responsive to velocity gradients in the y-direction, i.e. normal to the wall; then it becomes permissible to regard x and y as possessing different dimensions (viz. X and Y), with results which will be seen from the following table, wherein T stands for the dimension of time.

Table 1

Quantity	$d\delta/dx$	u_G	v_S	δ	du_G/dx	ν
Dimensions	Y/X	X/T	Y/T	Y	1/T	Y^2/T

The number of quantities appearing in equation (2) is 6; the number of independent dimensions is 3 (viz. X, Y and T). The π Theorem therefore leads to the conclusion that the corresponding dimensionally homogeneous equation must contain 6 − 3, i.e. 3, dimensionless quantities. Inspection, and some algebraic manipulation, shows that a suitable set of quantities is:

$$\frac{u_G\delta}{\nu}\frac{d\delta}{dx}, \quad \frac{\delta^2}{\nu}\frac{du_G}{dx}, \quad \text{and} \quad \frac{v_S\delta}{\nu}.$$

It will be found convenient to shift the first δ of the first group to the right of the differential sign. The dimensionally homogeneous form of equation (2) then becomes:

$$\frac{u_G}{\nu}\frac{d\delta^2}{dx} = F\left(\frac{\delta^2}{\nu}\frac{du_G}{dx}, \frac{v_S\delta}{\nu}\right) \tag{3}$$

where the function $F(\ldots)$ remains to be determined.

Discussion. It is evident that (3) is a first-order differential equation with δ^2 as the dependent variable and x as the independent variable. Of the other quantities, ν is a constant while u_G and v_S will be given, in any particular problem, as functions of x.† Given one boundary condition therefore, equation (3) can be solved in a particular case by well-known techniques of numerical analysis. Of course, to do this it is necessary to know the nature of the function $F(\ldots)$. Before embarking on the determination of $F(\ldots)$, it should be remarked that, according to our assumptions, this is a *general* function, valid for *all* two-dimensional laminar boundary layers. Because $F(\ldots)$ is a function of two variables, it can be represented by a family of curves on a single sheet of graph paper, or by a table with two arguments. We shall introduce such graphs and tables below.

When the boundary-layer thickness δ has been determined at every x by numerical solution of equation (3) for a particular case, it may be necessary to determine further properties, for example the local shear stress. This will be possible because the basic simplifying assumption (Section 2.2) implies that the desired property can be related to δ and the other local quantities by a further general function of three non-dimensional quantities; this will be exemplified below.

2.3. *The general functions obtained from the "similar" solutions*

The partial differential equations governing the laminar velocity boundary layer may be transformed, without approximation, to ordinary differential equations, when the free-stream velocity u_G obeys the differential equation:

$$\frac{du_G}{dx} = C\, u_G^n \tag{4}$$

where C and n are constants.‡ Since the velocity profiles in the y-direction are all similar to each

† Of course specification of u_G as a function of x automatically specifies du_G/dx as a function of x.

‡ An alternative definition which does not involve the constant C is: $u_G\,(d^2u_G/dx^2)/(du_G/dx)^2 = n$.

other in such a case, the corresponding solutions are known as the "similar" solutions.

Many such solutions are available. Their theory is discussed in Section 4 of this paper, and a survey of the solutions, together with new results, is presented in Paper 2 of the series. Their significance for the present section lies in the fact that the unknown functions referred to above may be deduced from them; the functions so deduced can thereafter be treated as generally valid, in accordance with the discussion in Section 2.2.

Definitions of boundary-layer thickness. Now that the main argument has been developed in general terms, it is time to introduce the definitions of the boundary-layer thicknesses which are of particular usefulness. These are:

Displacement thickness,

$$\delta_1 \equiv \int_0^\infty (1 - u/u_G)\,dy. \tag{5}$$

Momentum thickness,

$$\delta_2 \equiv \int_0^\infty (1 - u/u_G)\frac{u}{u_G}\,dy. \tag{6}$$

Shear thickness,

$$\delta_4 \equiv u_G/(\partial u/\partial y)_S. \tag{7}$$

where:

u is the velocity of the stream in the x-direction, the integrals are evaluated along a normal to the surface, and subscript S denotes the value in the fluid adjacent the surface.

The thickness ratios δ_1/δ_2 and δ_2/δ_4 are sometimes given special symbols, namely:

$$\delta_1/\delta_2 \equiv H_{12} \tag{8}$$

$$\delta_2/\delta_4 \equiv H_{24}. \tag{9}$$

One further symbol will be introduced at this stage, namely F_2. This is defined by:

$$\frac{u_G}{\nu}\left(\frac{d\delta_2^2}{dx}\right) \equiv F_2 \tag{10}$$

a definition which corresponds with equation (3), the boundary-layer thickness in question being δ_2, the momentum thickness.

The functions F_2, H_{12}, and H_{24} for the similar solutions. As is implicit in the dimensional analysis, the quantities F_2, H_{12} and H_{24} may be deduced from the similar solutions and expressed as functions of two variables. Since it is common practice† to choose δ_2 as the dependent variable of equation (3), we shall express F_2, H_{12} and H_{24} as functions of the arguments $(\delta_2^2/\nu)(du_G/dx)$ and $v_S\delta_2/\nu$.

The quantity F_2 is plotted as the ordinate in Fig. 2; $(\delta_2^2/\nu)(du_G/dx)$ is the abscissa and

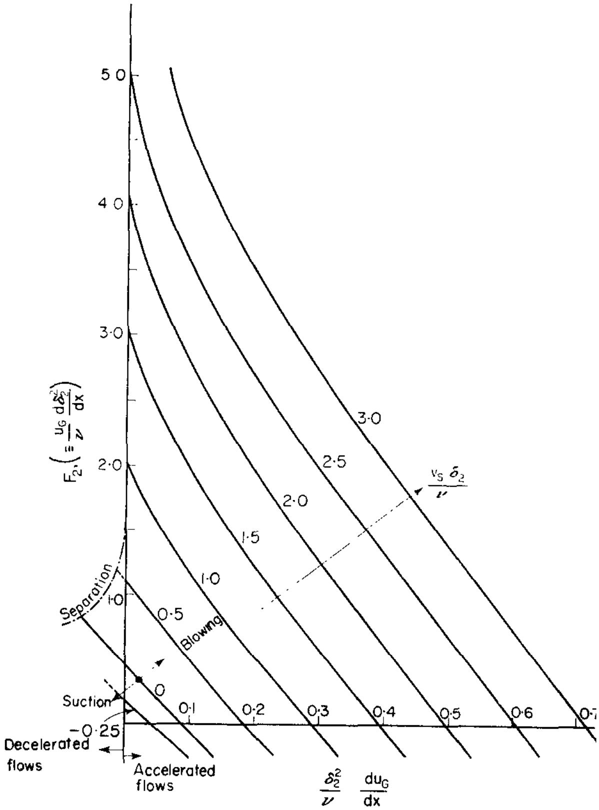

FIG. 2. Auxiliary function F_2 derived from the "similar" solutions (Evans).

$v_S\delta_2/\nu$ is the parameter. The curves have been obtained by Evans [1] using interpolation procedures described in Paper 2 of this series.

† The advantage is that then (3) is identical with the integral momentum equation. However one could also choose a different thickness, e.g. δ_1 or δ_4. The exploitation of this freedom of choice will be illustrated later when the method of Eckert [2] is described (Paper 4 of this series).

The latter paper also contains tables which embody the contents of Fig. 2 in a more accurate form.

Figure 3 contains a plot of the thickness ratio H_{24} plotted with the same abscissa and parameter,

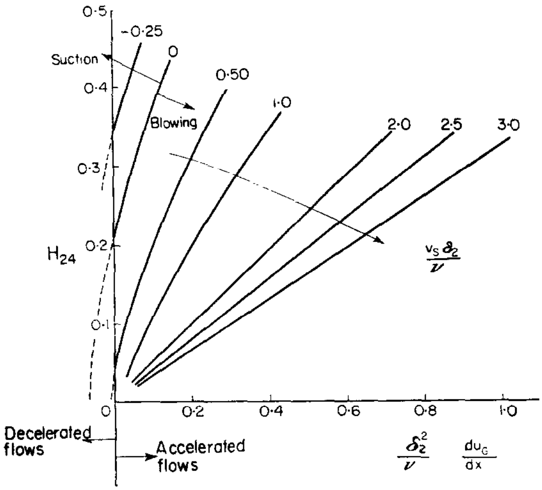

FIG. 3. Auxiliary function H_{24} deduced from the "similar" solutions (Evans).

and obtained from the same source. Fig. 4 contains a similar plot of the thickness ratio H_{12}. Tables of H_{24} and H_{12} may be found in Paper 2 of the present series.

The quantities F_2, H_{12}, H_{24}, $(\delta_2^2/\nu)(\mathrm{d}u_G/\mathrm{d}x)$ and $v_S\delta_2/\nu$ are linked by a simple relationship; for the integral momentum equation of the laminar boundary layer runs:

$$\frac{u_G}{\nu}\delta_2\frac{\mathrm{d}\delta_2}{\mathrm{d}x} + \left(2 + \frac{\delta_1}{\delta_2}\right)\frac{\delta_2^2}{\nu}\frac{\mathrm{d}u_G}{\mathrm{d}x} = \frac{\delta_2}{\delta_4} + \frac{v_S\delta_2}{\nu}. \quad (11)$$

On introduction of the definitions (8), (9) and (10), equation (11) becomes:

$$\tfrac{1}{2}F_2 + (2 + H_{12})\frac{\delta_2^2}{\nu}\frac{\mathrm{d}u_G}{\mathrm{d}x} = H_{24} + \frac{v_S\delta_2}{\nu}. \quad (12)$$

which is the relationship just referred to.

2.4. *Procedures for calculating the properties of the laminar boundary layer on a specified two-dimensional body*

We now consider how to solve the mathematical problem set out in Section 2.1.

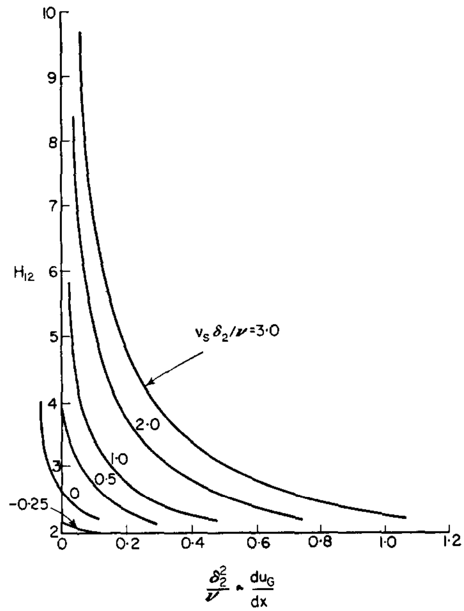

FIG. 4. Auxiliary function H_{12} deduced from the "similar" solutions (Evans).

Numerical analysis. We suppose that u_G and v_S are given as functions of x, and consider how to determine δ_2, δ_1, and the shear stress (via δ_4) at each value of x. The most accurate procedure is the following:

Step (i).

Choose reference values of velocity, U, and length, L; these may conveniently be the velocity far upstream of the body and the overall length along the surface of the body.

Step (ii).

Define non-dimensional quantities u_G^*, x^*, and δ^* as follows:

$$u_G^* \equiv u_G/U \quad (13)$$

$$x^* \equiv x/L \quad (14)$$

$$\delta^* \equiv \delta\sqrt{(U/L\nu)} \quad (15)$$

$$v_S^* \equiv v_S\sqrt{(L/\nu U)} \quad (16)$$

where the subscript to δ^* and δ may be 1, 2 or 4.

The differential equation to be solved then becomes:

$$u_G^*\frac{\mathrm{d}\delta_2^{*2}}{\mathrm{d}x^*} = F_2\left(\delta_2^{*2}\frac{\mathrm{d}u_G^*}{\mathrm{d}x^*},\, v_S^*\,\delta_2^*\right). \quad (17)$$

Step (iii).

Differentiate the $u_G^*(x^*)$ curve appropriate to the particular problem so that now du_G^*/dx^*, u_G^* and v_S^* are all available as functions of x^*.

Step (iv).

Solve equation (17) by standard procedures of numerical analysis (e.g. Runge–Kutta), referring to step (iii) for the functions particular to the problem, and to Fig. 2 (or corresponding tables) for values of the function F_2. (N.B. Note that $\delta_2^{*2}(du_G^*/dx^*)$ is identical in value with $(\delta_2^2/\nu)(du/dx)$, and that $v_S^*\delta_2^*$ is identical in value with $v_S\delta_2/\nu$.)

The result of this step is a curve of δ_2^* versus x^*.

Step (v).

Evaluate $\delta_2^{*2}(du_G^*/dx^*)$ at each x^*, and the corresponding values of H_{24} and H_{12} from Figs. 3 and 4 (or corresponding tables). Hence obtain δ_4^* and δ_1^* (if this is required).

Step (vi).

Evaluate the local shear stress at the wall, τ_S, from:

$$\tau_s g_0 = \mu \frac{u_G}{\delta_4} = \mu \frac{u_G}{\delta_4^*} \sqrt{\left(\frac{U}{L\nu}\right)} \tag{18}$$

The starting condition. Solution of a first-order differential equation requires the specification of an initial condition for the start of the integration. In the present problem this is given by the condition at the lowest value of x^*, i.e. at the upstream end of the boundary layer. x^* is usually defined to be zero here.

Three situations may arise:

Case (a). For a thin flat plate, δ_2^* is zero at the leading edge while u_G^* is finite there. Then both $\delta_2^{*2}(du_G^*/dx^*)$ and $v_S^*\delta_2^*$ are zero and F_2 is obtained from Fig. 2; the integration proceeds without difficulty.

Case (b). A body with a rounded leading edge has $u_G^* = 0$ at $x^* = 0$ while du_G^*/dx^* is finite. It follows that $u_G^*(d\delta_2^{*2}/dx^*)$ and therefore F_2 are zero at the leading edge.

The condition $F_2 = 0$ implies a relation between $\delta_2^{*2}(du_G^*/dx^*)$ and $v_S^*\delta_2^*$ (obtained by inspecting the abscissa scale of Fig. 2). Since both du_G^*/dx^* and v_S^* are known at $x^* = 0$, a simple graphical procedure enables δ_2^* to be found from the above-mentioned relationship. The numerical integration can then proceed.

Case (c). Sometimes we have: $u_G^* = 0$ at $x^* = 0$ and $u_G^* = Kx^{*Eu}$ in the immediate neighbourhood, where K and Eu are known constants. It can be shown† that, in this case,

$$u_G^* \frac{d\delta_2^{*2}}{dx^*} = \frac{1 - Eu}{Eu} \cdot \delta_2^{*2} \frac{du_G^*}{dx^*}. \tag{19}$$

Equation (19) represents a straight line passing through the origin of Fig. 2 with slope $(1 - Eu)/Eu$; it thus specifies a relation between $\delta_2^{*2}(du_G^*/dx^*)$ and $v_S^*\delta_2^*$, which can be solved for δ_2^* in the same manner as in case (b). (N.B. Either $v_S^* = 0$, or $v_S^*\delta_2^* =$ *a known value*, in practical problems. The problem is therefore easier than that of case (b)).

Quadrature procedure. The curves on Fig. 2 are approximately straight. The straightness of the line $v_S\delta_2/\nu = 0$ was noticed by Walz [14] and Thwaites [12], and utilized by them as the foundation of a quadrature procedure. We can generalize their procedure in a way which is particularly useful if $v_S\delta_2/\nu$ happens to be constant over the surface in question, but has also a wider application, by putting:

$$E_2 \equiv F_2 + a(\delta_2^2/\nu)(du_G/dx) - d \tag{20}$$

where a and d are constants. Then equation (17) becomes:

$$u_G^* \cdot \frac{d\delta_2^{*2}}{dx^*} + a\delta_2^{*2}\left(\frac{du_G^*}{dx^*}\right) = E_2 + d \tag{21}$$

wherein of course E_2 is a non-linear function of $\delta_2^{*2}(du_G^*/dx^*)$ and of $v_S^*\delta_2^*$, which may be regarded as the "error" involved in the straight-line approximation.

If equation (21) is first multiplied through by u_G^{*a-1}, it can be written in integrated form as:

$$\delta_2^{*2} u_G^{*a} = \int_0^{x^*} (d + E_2) u_G^{*a-1}\, dx^* \tag{22}$$

wherein the starting condition $\delta_2^* = 0$ or $u_G^* = 0$ at $x^* = 0$ has already been inserted. It should be noted that equation (22) is exact, if equation (17) is regarded as exact.

Part of the right-hand side of (22), namely $\int_0^{x^*} du_G^{*a-1}\, dx^*$ can be evaluated by quadrature,

† See equation (70) below.

since d is a constant and u_G^* is a known function of x^*; the other part, namely $\int_0^{x^*} E_2\, u_G^{*a-1}\, dx^*$ cannot be so evaluated, since E_2 depends on δ_2^* as well as on x^*. However the second part can be determined in an iterative procedure in which a first approximation to δ_2^*, namely $\delta_{2,\mathrm{I}}^*$ is evaluated from the formula:

$$\delta_{2,\mathrm{I}}^* = \left[\frac{\int_0^{x^*} d u_G^{*a-1}\, dx^*}{u_G^{*a}}\right]^{1/2} \tag{23}$$

this value of δ_2^* is then used to give a first approximation to the value of E_2 at each x^*, namely $E_{2,\mathrm{I}}$, by reference to appropriate tables (see Table 2 for example).

Table 2

$\frac{\delta_2^2}{\nu}\frac{du_G}{dx}$	E_2	$\frac{\delta_2^2}{\nu}\frac{du_G}{dx}$	E_2	$\frac{\delta_2^2}{\nu}\frac{du_G}{dx}$	E_2
−0·0681	−0·0280	0·0854	0·0	0·1275	−0·0049
−0·0580	−0·0195	0·0963	−0·0013	0·1307	−0·0052
−0·0265	−0·0050	0·1036	−0·0019	0·1327	−0·0054
0·0	0·0	0·1065	−0·0025	0·1370	−0·0062
0·0333	0·0025	0·1109	−0·0023	0·1415	−0·0067
0·0538	0·0018	0·1160	−0·0033	0·169	−0·0105
0·0677	0·0011	0·1215	−0·0042	0·385	−0·007
0·0778	0·0003				

Thereupon a second approximation, $\delta_{2,\mathrm{II}}$, can be evaluated from:

$$\delta_{2,\mathrm{II}}^* = \left[\frac{\int_0^{x^*} d u_G^{*a-1}\, dx^* + \int_0^{x^*} E_{2,\mathrm{I}}\, u_G^{*a-1}\, dx^*}{u_G^{*a}}\right]^{1/2} \tag{24}$$

and so on until $\delta_2^*(x^*)$ has been obtained to sufficient accuracy.

This procedure will lead to the correct solution of the differential equation whatever values are chosen for the constants a and d. However a judicious choice of these quantities will make E_2, which may be regarded as a "correction" term, very small, and so greatly reduce the number of iterations which are needed. Indeed Walz and Thwaites, who were concerned only with the case of $v_S^*\delta_2^* = 0$, recommended the values: $a = 6{\cdot}10$, $d = 0{\cdot}47$ (Walz), or $a = 6$, $d = 0{\cdot}45$ (Thwaites), and $E_2 = 0$; i.e. they proposed that only the first approximation, equation (23), should be used and did not consider further iteration. The error is small because of the straightness of the curve $v_S\delta_2/\nu = 0$ on Fig. 2.

A somewhat improved procedure for the same case of no mass transfer has been proposed by the present author [10]. The constants a and d are chosen so that the corresponding straight line passes through the intersections of the curve $v_S\delta_2/\nu = 0$ with the co-ordinate axes of Fig. 2; the values are: $a = 5{\cdot}164$, $d = 0{\cdot}441$. The corresponding values of E_2 are contained in the Table 2, compiled by Evans [1].

The corresponding procedure for mass transfer involves making estimates of the quantity $v_S^*\delta_2/\nu$ (i.e. $v_S^*\delta_2^*$) at the stagnation point and at the point where du_G/dx is zero, and drawing a straight line on Fig. 2 to pass through the appropriate points on the co-ordinate axes. The magnitude of E_2 will then be small in most practical cases, so that usually the second approximation, given by equation (24), will be sufficiently accurate.

It should be borne in mind that the equation which we are solving, namely equation (17), itself only approximately represents the full partial differential equation; the attempt to obtain its solution to an accuracy of better than, say, 2 per cent would therefore ordinarily represent wasted effort.

2.5 *The axi-symmetrical case*

The same techniques may be used for calculating boundary-layer growth and wall friction when the body and the surrounding stream are axi-symmetrical (Fig. 5). In this case, specification of the problem must also include radius, R, as a function of x, in addition to the stream velocity u_G and the normal velocity at the wall v_S.

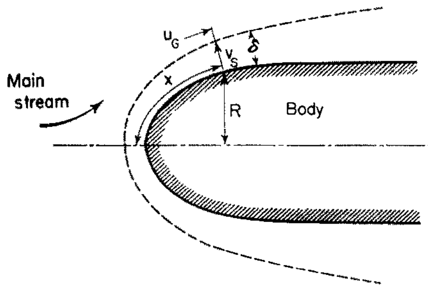

FIG. 5. Laminar boundary layer on an axi-symmetrical body.

The Mangler transformation. Mangler [6] has shown that the partial differential equations of the axi-symmetrical laminar boundary layer may be transformed so as to have the same form as those of the two-dimensional boundary layer. The implication for the present methods is that the non-dimensional quantities used in solving the equation of boundary-layer growth should differ slightly from those of equations (13) to (16). Specifically, we should define:

$$u_G^{**} \equiv u_G/U \tag{25}$$

$$x^{**} \equiv (1/L) \int_0^x (R/R_0)^2 \, dx \tag{26}$$

where R_0 is some reference radius. (We could take $R_0 = L$ for example),

$$\delta^{**} = \delta(R/R_0) \sqrt{(U/L\nu)} \tag{27}$$

$$v_S^{**} = v_S (R_0/R) \sqrt{(L/\nu U)}. \tag{28}$$

The differential equation to be solved then becomes:

$$u_G^{**} \frac{d\delta_2^{**2}}{dx^{**}} = F_2 \left(\delta_2^{**2} \frac{du_G^{**}}{dx^{**}}, \; v_S^{**} \delta_2^{**} \right) \tag{29}$$

where u_G^{**} and v_S^{**} are given functions of x^{**}, and F_2 is the same function as before despite the increased bespanglement of its arguments.

Comparison of the definitions (13) to (16) with definitions (25) to (28) shows that:

$$\delta_2^{**2} \frac{du_G^{**}}{dx^{**}} = \delta_2^{*2} \frac{du_G^*}{dx^*} = \frac{\delta_2^2}{\nu} \frac{du_G}{dx} \tag{30}$$

$$v_S^{**} \delta_2^{**} = v_S^* \delta_2^* = \frac{v_S \delta_2}{\nu} \tag{31}$$

while:

$$\left.\begin{aligned} & u_G^{**} \frac{d\delta^{**2}}{dx} \\ & = u_G^* \left(\frac{R_0}{R}\right)^2 \frac{d\{(R/R_0)\,\delta^*\}^2}{dx^*} \\ & = \frac{u_G}{\nu} \left(\frac{R_0}{R}\right)^2 \frac{d\{(R/R_0)\,\delta\}^2}{dx} \\ & = \frac{u_G}{\nu} \frac{d\delta^2}{dx} + \frac{u_G \delta^2}{\nu} \left(\frac{R_0}{R}\right)^2 \frac{d(R/R_0)^2}{dx} \end{aligned}\right\} \tag{32}$$

Solution of the equation. Solution of (29) by numerical integration does not require any special remark, except perhaps that the quadrature of equation (26) needs to be carried out as a preliminary so that the functions: $u_G(x)$, $v_S(x)$ and $R(x)$ can be transformed into the functions: $u_G^{**}(x^{**})$ and $v_S^{**}(x^{**})$.

If an iterative quadrature procedure is to be used for solving the differential equation, as described above, the preliminary step just mentioned can be incorporated in the main quadrature; indeed the doubly starred quantities need not then appear explicitly. Thus by writing equation (22) in terms of δ_2^{**}, x^{**}, etc., and then transforming back to the singly starred quantities, we obtain an equation which may be written as:

$$\delta_2^* = \left[\frac{\int_0^{x^*} (d + E_2)\,(R/R_0)^2\, u_G^{*a-1}\, dx^*}{(R/R_0)^2\, u_G^{*a}} \right]^{1/2} \tag{33}$$

in which E_2 is a function of $\delta_2^{*2}(du_G^*/dx^*)$ and $v_S^*\delta_2^*$. This equation may be solved by iteration as before, the number of cycles which are needed depending on the choice of constants d and a.

The axi-symmetrical stagnation point. The forward stagnation point of an axially symmetrical body is characterized by the following conditions: $u_G = 0$, $du_G/dx =$ finite, $R = x$. Consequently,

$$u_G^{**} \propto x^{**1/3}. \tag{34}$$

This may be proved as follows:

Inserting $R = x$ in equation (26), and integrating, we obtain:

$$x^{**} = \frac{1}{3}\,\frac{x^3}{R^2L}, \tag{35}$$

The conditions $u_G = 0$, $\mathrm{d}u_G/\mathrm{d}x \neq 0$, on the other hand, dictate that:

$$u_G^{**} = x^{**} \cdot \frac{\mathrm{d}u_G^{**}}{\mathrm{d}x^{**}}$$

$$= x^{**}\,\frac{L}{U}\left(\frac{R_0}{x}\right)^2 \frac{\mathrm{d}u_G}{\mathrm{d}x}. \tag{36}$$

Combining equations (35) and (36), we obtain:

$$u_G^{**} = x^{**1/3} \cdot \left(\frac{R_0}{3L}\right)^{2/3} \frac{L}{U}\,\frac{\mathrm{d}u_G}{\mathrm{d}x}. \tag{37}$$

The exponent of x^{**} appearing in equations (34) and (37) indicates that, for the reasons outlined in connexion with equation (19), the axially symmetrical stagnation point is represented by a line of slope $(1 - 1/3)/(1/3)$, i.e. of 2, on Fig. 2.

3. EXAMPLES OF THE USE OF THE METHODS

The main purpose of the present section is to illustrate the use of the methods which have been described. The opportunity will be taken to consider some examples for which exact solutions are available to the partial differential equations of the boundary layer: comparison of these solutions with those given by the present approximate methods will give an indication of the accuracy of the latter.

3.1. *Uniformly decelerating flow; no mass transfer*

The problem. Howarth [3] has considered the growth of a two-dimensional laminar boundary layer on a body when the main-stream velocity varies according to the law:

$$u_G = U(1 - x/L) \tag{38}$$

where U and L are constants. He found that the shear stress on the wall falls to zero, i.e. the boundary layer separates from the wall, at a finite distance from the leading edge, viz. x_{sep}, where

$$x_{sep} \approx 0{\cdot}12L. \tag{39}$$

Quadrature method; first approximation. Equation (23) combined with (38) leads to:

$$\delta_{2,\mathrm{I}}^{*} = \left[\frac{\int_0^{x^*} d(1-x^*)^{a-1}\,\mathrm{d}x^*}{(1-x^*)^a}\right]^{1/2}$$

$$= \left(\frac{d}{a}\right)^{1/2}\left[\frac{1}{(1-x^*)^a} - 1\right]^{1/2} \tag{40}$$

Now if the shear stress is zero, then $H_{24} = 0$. This occurs, according to Fig. 3, when $(\delta_2^2/\nu)\,(\mathrm{d}u_G/\mathrm{d}x) = -0{\cdot}0681$. Since

$$\mathrm{d}u_G/\mathrm{d}x = -\,U/L$$

in the present case, δ_2^* for separation is equal to $(0{\cdot}0681)^{1/2}$. Inserting this conclusion in equation (40), we find:

$$x_{sep}^* = 1 - 1/\left[1 + 0{\cdot}0681\,\frac{a}{d}\right]^{1/a}. \tag{41}$$

x_{sep}^* may now be evaluated by a suitable choice of a and d. The three sets of values encountered so far give results tabulated in Table 3.

Table 3

a	d	Proposed by	x_{sep}^* from equation (41)
6·10	0·47	Walz	0·0985
6·0	0·45	Thwaites	0·1021
5·164	0·441	Spalding	0·1073

The value for x_{sep}^* calculated by Howarth is 0·12 approximately [9]. Comparison of this value with those of Table 3 shows that each of the values obtained from equation (41) is an under-estimate, the error being between 10 and 20 per cent. If higher approximations are taken for δ_2^*, as described in Section 2.4, differences between the values of x_{sep}^* based on the various choices of a and d naturally disappear; however this iterative procedure converges to a value of x_{sep}^* which is still about 10 per cent less than Howarth's value. This error is presumably due, in the main, to the approximation inherent in the present method; however the accuracy of Howarth's value is also somewhat uncertain.

3.2. *The flat plate with uniform suction*

The problem. Iglisch [16] has considered the growth of a two-dimensional laminar boundary layer on a flat plate immersed in a stream of uniform velocity, when mass transfer occurs at a rate which is uniform over the plate surface; the direction of mass transfer is from the fluid to the plate.

He found that the boundary layer thickness does not grow indefinitely, but reaches an asymptotic value; correspondingly the shear stress falls to a steady finite value. We shall compare the predictions of the present approximate method with those of Iglisch.

Numerical analysis. Since, in this case, u_G is independent of x, F_2 can be regarded as a function of $(v_S\delta_2/\nu)$ alone. Equation (10) now becomes:

$$\frac{u_G}{\nu}\frac{\mathrm{d}\delta_2^2}{\mathrm{d}x} = F_2(v_S\delta_2/\nu) \tag{42}$$

where the values of F_2 are obtained from the ordinate axis of Fig. 2.

Since there is no obvious reference velocity, U, or length, L, in this problem, we shall not introduce the starred quantities. Equation (42) is therefore re-arranged immediately to yield:

$$\frac{v_S^2 x}{u_G \nu} = \int_0^{\frac{v_S\delta_2}{\nu}} \frac{\mathrm{d}(v_S\delta_2/\nu)^2}{F_2}. \tag{43}$$

Thus the present problem reduces to a quadrature without the need for any linear hypothesis.

Equation (43) leads, on numerical evaluation, to the relation between $v_S\delta_2/\nu$ and $v_S^2 x/u_G\nu$ represented by the upper curve of Fig. 6. The lower curve is that obtained by Iglisch by way of exact numerical solution of the partial differential equation of the boundary layer. It will be seen that the agreement is within about 2 per cent.

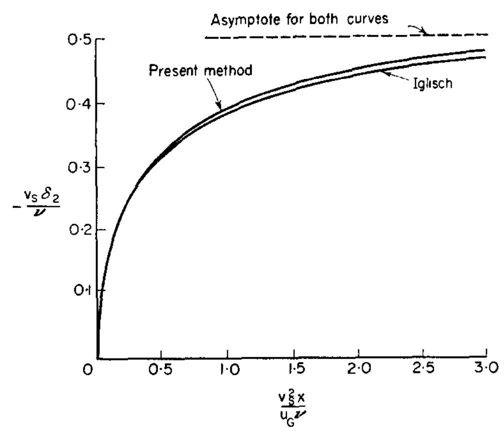

FIG. 6. Growth of momentum thickness on a flat plate with uniform suction.

3.3. *The axi-symmetrical stagnation point*

The problem. Axi-symmetrical bodies, such as missiles, fuel droplets, etc., have a stagnation point on the leading face. The flow in this region is frequently laminar. Missile noses may be transpiration-cooled; fuel droplets have an appreciable rate of mass transfer across their surfaces: the influence of a finite mass transfer rate on the surface shear stress near a stagnation point is therefore of interest.

Solution using Figs. 2 *and* 3. As explained in Section 2.5, conditions at an axially symmetrical stagnation point can be represented, as a relation between $u_G^{**}\mathrm{d}\delta_2^{**}/\mathrm{d}x^{**}$ versus $v_S^{**}\delta_2^{**}$, by a line at a slope of 2 on Fig. 2. This relation may be transformed into physical terms by way of equations (31) and (32), putting $R = x$. Fig. 7

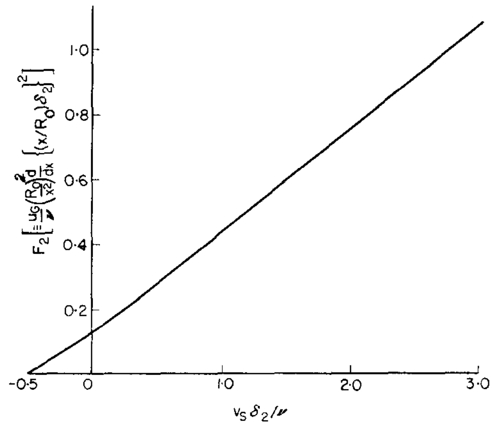

FIG. 7. Auxiliary function F_2 for the axi-symmetrical stagnation point. ($\beta = 0{\cdot}5$).

shows the resulting relation. Negative values of $v_S\delta_2/\nu$ have been included though they are of infrequent practical importance.

Now, as indicated on Fig. 7,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{R_0}\delta_2\right)^2 = F_2\frac{\nu}{u_G}\left(\frac{x}{R_0}\right)^2$$

$$= F_2\frac{\nu}{\mathrm{d}u_G/\mathrm{d}x}\cdot\frac{x}{R_0^2}. \tag{44}$$

Hence we obtain, by integration, an expression for δ_2:

$$\left(\frac{x}{R_0}\,\delta_2\right)^2 = \frac{F_2\nu\, x^2}{2R_0^2\,(\mathrm{d}u_G/\mathrm{d}x)}$$

$$\text{i.e. } \delta_2^2 = \frac{F_2\nu}{2(\mathrm{d}u_G/\mathrm{d}x)}. \tag{45}$$

Equation (45) can be used, in conjunction with Fig. 7 and equation (18), to yield momentum thickness, displacement thickness and wall shear stress as functions of the mass transfer rate, all these quantities being in dimensionless form. Fig. 8 illustrates the results.

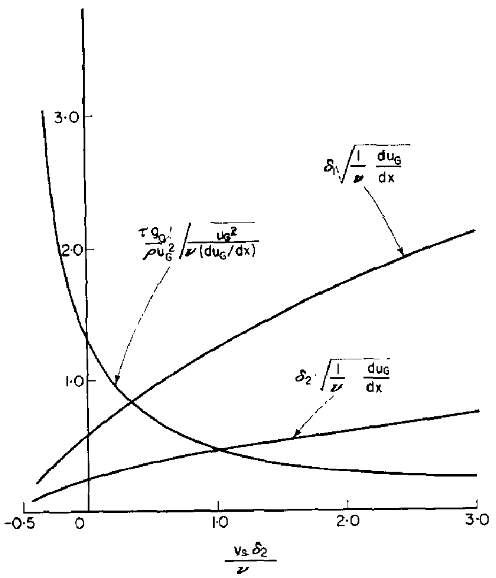

FIG. 8. Boundary layer properties at the axi-symmetrical stagnation point, as functions of mass transfer rate.

Discussion. The physical significance of Fig. 8 is easy to discern: an outward rate of mass transfer ($v_S > 0$) increases the boundary-layer thicknesses and decreases the shear stress; an inward rate of mass transfer has the opposite effects.

It should be noted that the results contained in Fig. 8 represent *exact* solutions of the partial differential equation of the boundary layer; for the axi-symmetric stagnation point has a stream velocity distribution obeying equation (4), so that the data of Figs. 2, 3 and 4 are exact.

3.4. *The accuracy and convenience of the method*

The above examples have shown that the method presented here can be used to provide predictions of the distribution of boundary-layer thickness and surface shear; these are fairly close to the predictions obtained by more exact and time-consuming methods. The difference between the two predictions is less than the experimental error usually associated with boundary-layer studies.

However, it should be emphasized that the examples chosen have all been particularly easy ones: the numerical integration of equation (17) cannot always be escaped. Nevertheless the method presented is very much easier than the exact numerical solution of the partial differential equation which it has supplanted.

4. THE ORIGIN OF FIGS. 2, 3 AND 4: "SIMILAR" SOLUTIONS

4.1. *Introduction*

In this section we shall explain the origin of the functions F_2, H_{12} and H_{24} which form an essential part of the present method. The section is included partly for the convenience of readers interested in the fundamentals of the subject, and partly because the treatment involves some departures from that which has become conventional (see, for example, that of Schlichting [9]).

The novelties have been introduced to avoid some of the less elegant features of the conventional treatment. These include the following:

(i) It is common to represent the "similar" boundary layers as those which would be formed on wedges of angle β. However, since the β-values of the solutions in which we are interested range from $+\infty$ to $-\infty$, whereas the angle of a physically realizable wedge must fall in a more restricted range, the interpretation in terms of wedges may hamper understanding. Moreover, it is entirely accidental that the solutions for inviscid flow near wedges give velocity distributions corresponding to equation (4): if such velocity distributions were instead given by much more complex shapes of body the boundary-layer analyses would in no way be altered.

(ii) It is also common to represent the "similar" boundary layers as those which prevail when the free stream velocity obeys the law: $u_G = const.\ x^{Eu}$, where Eu is a constant known as the Euler number. The disadvantages of this procedure are (a) that one member of the family of similar solutions, namely that for a free stream velocity distribution according to $u_G = const.\ e^{const.\ x}$, is thereby excluded, and (b) that attention is concentrated on the length x which in some cases is the distance *downstream* of the leading edge while in others it is the distance *upstream* of some singularity in the free-stream flow.

(iii) The fundamental ordinary differential equations satisfied by the similar solutions is sometimes written and discussed in terms of two constants α and β. This falsely gives the impression that a two-parameter family of differential equations is in question. It will be shown below that the whole range of solutions can be discussed by reference to a single parameter, for example β.

The characteristic of the free-stream velocity distributions which leads to the existence of the similar solutions is chosen, in the present treatment, as follows: u_G must obey the differential equation:

$$\frac{du_G}{dx} = Cu_G^n \tag{46}$$

where C and n are constants. It will be noted that this equation makes reference to dx, but not to x by itself. The solutions also therefore refer only to dx. This is particularly desirable when the "similar" solutions are to be used, as in the present paper, as aids in the solving of "non-similar" problems; for, at *corresponding* points in the "similar" and "non-similar" boundary layers, x has a different value for each of the layers.

4.2. *The partial differential equations of the boundary-layer*

We shall here consider only the two-dimensional boundary layer; Section 4.5 deals with the transformation by which the equations for the axi-symmetrical boundary layer are transformed so as to have the same form as those of the two-dimensional boundary layer.

The equation of motion of the fluid can be written, when the properties are uniform, as:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_G\frac{du_G}{dx} + \nu\frac{\partial^2 u}{\partial y^2}. \tag{47}$$

The continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{48}$$

These equations are commonly combined by introducing the stream-function ψ, defined by:

$$u \equiv \frac{\partial \psi}{\partial y};\ v = -\frac{\partial \psi}{\partial x}. \tag{49}$$

There results:

$$\frac{\partial \psi}{\partial y}\cdot\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\cdot\frac{\partial^2 \psi}{\partial y^2} = u_G\frac{du_G}{dx} + \nu\frac{\partial^3 \psi}{\partial y^3}. \tag{50}$$

The first novel step of the present treatment will now be made by eliminating dx from the equation so that the independent variables become y and u_G. There results

$$\frac{\partial \psi}{\partial y}\cdot\frac{\partial^2 \psi}{\partial u_G \partial y} - \frac{\partial \psi}{\partial u_G}\cdot\frac{\partial^2 \psi}{\partial y^2} = u_G + \nu\frac{\partial^3 \psi}{\partial y^3}\cdot\frac{dx}{du_G} \tag{51}$$

Equation (51) is generally valid. We now restrict consideration to the flows leading to similar boundary layers by introducing (46) into (51). The partial differential equation which finally has to be solved is thus:

$$\nu\frac{\partial^3 \psi}{\partial y^3} = Cu_G^n\left[\frac{\partial \psi}{\partial y}\cdot\frac{\partial^2 \psi}{\partial u_G \partial y} - \frac{\partial \psi}{\partial u_G}\cdot\frac{\partial^2 \psi}{\partial y^2} - u_G\right] \tag{52}$$

4.3. *The ordinary differential equation of the similar boundary layer*

The solution of equation (52) involves finding the value of ψ for each pair of values of the independent variables y and u_G, and with ν,

C and n specified. Vectorial dimensional analysis of this situation along the lines explained in Section 2.2 above, reveals that the required relation may be reduced to one of the form:

$$\frac{\psi}{u_G}\sqrt{\left(\frac{Cu_G{}^n}{\nu}\right)} = f\left\{y\sqrt{\left(\frac{\nu}{Cu_G{}^n}\right)}, n\right\}. \quad (53)$$

Since the quantity n is a numerical constant, equation (53) is a relation between *two* variables: the corresponding differential equation must therefore be an *ordinary* one, not a partial. We are thus assured by dimensional analysis that equation (52) will reduce to an ordinary differential equation if we introduce the transformations:

$$\zeta \equiv \frac{\psi}{u_G}\sqrt{\left(\frac{Cu_G^n}{\nu}\right)} \quad (54)$$

$$\xi \equiv y\sqrt{\left(\frac{Cu_G^n}{\nu}\right)}. \quad (55)$$

These transformations lead, by straightforward mathematical manipulations, to:

$$\frac{d^3\zeta}{d\xi^3} + (1 - n/2)\,\zeta\,\frac{d^2\zeta}{d\xi^2} + 1 - \left(\frac{d\zeta}{d\xi}\right)^2 = 0. \quad (56)$$

The equation is now in a form in which it can be discussed mathematically. However, in order to make it familiar to readers conversant with the standard treatment, we multiply both ζ and ξ by the quantity $(1 - n/2)^{1/2}$ and replace $(1 - n/2)$ by $1/\beta$. With the corresponding definitions:

$$\beta \equiv 1/(1 - n/2) \quad (57)$$

$$f \equiv (1 - n/2)^{1/2}\zeta \quad (58)$$

$$= \frac{\psi}{u_G}\sqrt{\left(\frac{du_G}{dx}\frac{1}{\beta\nu}\right)} \quad (58)$$

$$\eta \equiv (1 - n/2)^{1/2}\xi$$

$$= y\sqrt{\left(\frac{1}{\nu\beta}\frac{du_G}{dx}\right)} \quad (59)$$

equation (56) now takes up the familiar form:

$$f''' + ff'' + \beta(1 - f'^2) = 0 \quad (60)$$

where the primes denote differentiation with respect to η.

In the f and η variables, reference to equation (49) shows that:

$$f' = u/u_G \quad (61)$$

and

$$f_0 = -v_S/\sqrt{\left(\frac{\nu}{\beta}\frac{du_G}{dx}\right)} \quad (62)$$

suffix zero indicating the value when $\eta = 0$.

The boundary conditions corresponding to the situations discussed in the present paper are:

At the wall, the velocity u equals zero. Therefore

$$\eta = 0: \; f' = 0. \quad (63)$$

At a large distance from the wall, the velocity u equals the free-stream velocity u_G. Thus

$$\eta = \infty: \; f' = 1. \quad (64)$$

At the wall again, the normal velocity has the prescribed value v_S. Therefore:

$$f = f_0 = -v_S/\sqrt{\left(\frac{\nu}{\beta}\frac{du_G}{dx}\right)}. \quad (65)$$

Incidently, it should be noted that, for a similar boundary layer to exist, equations (65) and (46) together imply that the velocity v_S must obey the relation: $v_S \propto u_G^{n/2}$; for f_0 must be constant.

4.4. *Discussion of the family of similar solutions*

Physical significance. Equation (60), together with boundary conditions (63), (64) and (65), define a family of $f - \eta$ relations with two parameters; the latter may conveniently be β and f_0. Solutions with positive f_0 correspond to negative v_S, i.e. to mass transfer processes such as condensation or absorption of one component from the mixture under consideration; solutions with negative f_0 correspond to mass transfer processes with positive v_S, such as vaporization or transpiration-cooling.

The significance of β may be studied by considering its appearance in equations (58) and (59). If we were only concerned with mathematically real values of f and η, the expressions within the square-root expressions would have to be positive. Now ν is always positive; β would therefore be positive when du_G/dx is positive, and negative when du_G/dx is negative. Thus $\beta > 0$ would signify an accelerating flow; $\beta < 0$ signifies a decelerating flow. $\beta = 0$ then would signify a flow of uniform velocity, though this is less obvious.†

However, imaginary values of f and η are also of interest, since it is possible for physical situations to arise in which β and du_G/dx have opposite signs. This is seen most clearly by reference to equation (4), for which there is no physical restriction on either the magnitude of n or the sign of C. Equations (58) and (59) insure that if f is imaginary then so is η, and vice versa; for the physical quantities u, ψ, etc., must be real. This means that f' is always real, as is demanded by equation (61), and v_S also.

It follows that we shall be interested in obtaining solutions to the mathematical problem for both positive and negative values of β and for both real and imaginary positive and negative values of f_0.

Methods of presenting the solutions. The solutions of equation (60) for a given pair of values of f_0 and β may conveniently be expressed in the following terms:

(i) A curve f' versus η. This represents the velocity distribution along a normal to the wall, as indicated by equation (61).

(ii) A value f_0''. This represents the velocity gradient at the wall, $[(\partial u/\partial y)]_S$, to which it is related by:

$$f_0'' = \frac{1}{u_G}\left(\frac{\partial u}{\partial y}\right)_S \cdot \left\{\frac{\beta \nu}{du_G/dx}\right\}^{1/2} = \frac{1}{\delta_4}\left(\frac{\beta \nu}{du_G/dx}\right)^{1/2}. \tag{66}$$

(iii) A value of the dimensionless displacement thickness, $\int_0^\infty (1 - f')d\eta$. The definitions introduced above show that this is related to the displacement thickness δ_1 by:

$$\int_0^\infty (1 - f')d\eta = \delta_1 \left(\frac{1}{\beta\nu}\frac{du_G}{dx}\right)^{1/2} = \frac{1}{\beta^{1/2}}\left(\frac{\delta_1^2}{\nu}\frac{du_G}{dx}\right)^{1/2}. \tag{67}$$

(iv) A value of the dimensionless momentum thickness, $\int^\infty f'(1 - f')d\eta$. The definitions introduced above show that this is related to the momentum thickness δ_2 by:

$$\int_0^\infty f'(1 - f')d\eta = \delta_2\left(\frac{1}{\beta\nu}\frac{du_G}{dx}\right)^{1/2} = \frac{1}{\beta^{1/2}}\left(\frac{\delta_2^2}{\nu}\frac{du_G}{dx}\right)^{1/2}. \tag{68}$$

It will be noted that equation (68) contains explicitly one of the quantities which have been used in Sections 2 and 3 of this paper.

(v) A value of F_2, i.e. of $(u_G/\nu)d\delta_2^2/dx$. This may be obtained from (68) via the relation:

$$\frac{u_G}{\nu}\frac{d\delta_2^2}{dx} = 2\left(\frac{1}{\beta} - 1\right)\frac{\delta_2^2}{\nu}\frac{du_G}{dx} \tag{69}$$

which is proved by recognizing that, for a similar boundary layer:

$$\frac{\delta_2^2}{\nu}\frac{du_G}{dx} = \text{const.};$$

so

$$\frac{\delta_2^2}{\nu} . Cu_G^n = \text{const.};$$

therefore

$$\frac{d\delta_2^2}{\delta_2^2} + n\frac{du_G}{u_G} = 0. \tag{70}$$

(vi) Values of H_{12}, i.e. of δ_1/δ_2, and of H_{24}, i.e. of δ_2/δ_4. These are derived from the above results in a straightforward manner.

(vii) Values of $v_S\delta_2/\nu$. These may clearly be obtained, by reason of equations (65) and (68), from the relation:

$$\frac{v_S\delta_2}{\nu} = -f_0\int_0^\infty f'(1 - f')d\eta. \tag{71}$$

The above quantities need not be presented as functions of f_0 and β as has just been suggested: instead one may take *any* two quantities derived from the solution and express the other quantities as functions of them. Figs. 2, 3 and 4 furnish

† This is best seen by introducing x in this instance. For it is easily shown that (for $n \neq 1$):

$$du_G/dx = (u_G/x)/(1 - n).$$

So du_G/dx tends to zero as $n \to \infty$. But so does β according to equation (57). Thus for the flat plate (u_G = const.). $(1/\beta)$ (du_G/dx) becomes indeterminate and must be replaced by $(1/2)$ (u_G/x).

examples of the latter practice: there F_2, H_{24} and H_{12} are respectively represented as functions of the independent variables $(\delta^2/\nu)\,(du_G/dx)$ and $v_S\delta_2/\nu$. Other re-arrangements are obviously possible. Sometimes the Euler number is used as a parameter; it is related to n and β, according to the above definitions, by:

$$Eu = 1/(1-n) = \beta/(2-\beta). \qquad (72)$$

Obtaining the solutions. The method by which equation (60) is solved in order to yield the above results does not concern us here. It will merely be remarked that the equation is non-linear and possesses boundary conditions at both ends of the range of integration; inevitably iterative numerical techniques have to be used. These involve considerable labour.

The difficulty of obtaining solutions to equation (60) is such that an insufficient number have been obtained. However, for carrying out the computations discussed in Sections 2 and 3 of this paper, complete solutions of equation (60) are not needed: the $f' - \eta$ curve, for example, is not required. It suffices to possess plots such as shown in Figs. 2, 3 and 4.

Although the available exact solutions are only sparsely scattered over the areas of interest in these diagrams, they can serve as foundation-points around which extensive charts can be constructed by interpolation. The means of doing so are described in Paper 2 of the present series.

4.5. *Axi-symmetrical flows*

For completeness we give here the details of the Mangler transformation by which the partial differential equations of the axi-symmetrical laminar boundary layer can be transformed into those of the two-dimensional boundary layer.

For the axi-symmetrical case, the momentum and continuity equations are respectively:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_G\frac{du_G}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (73)$$

and

$$\frac{\partial}{\partial x}(Ru) + \frac{\partial}{\partial y}(Rv) = 0 \qquad (74)$$

Mangler [6] introduces the following changes of variables:

$$d\bar{x} = \left(\frac{R^2}{R_0^2}\right)dx \qquad (75)$$

$$\bar{y} = \left(\frac{R}{R_0}\right)y \qquad (76)$$

$$\bar{v} = \frac{R_0}{R}\left(v + y\frac{u}{R}\frac{dR}{dx}\right) \qquad (77)$$

$$\bar{u} = u. \qquad (78)$$

Equations (73) and (74) then transform to:

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \bar{u}_G\frac{d\bar{u}_G}{d\bar{x}} + \nu\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \qquad (79)$$

and

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\bar{x}\partial} = 0. \qquad (80)$$

These will be recognized as possessing the same form as equations (47) and (48), valid for the two-dimensional boundary layer.

It should be clear that the transformation introduced in Section 2.5 represents a particular form of that just discussed. It is the existence of the transformation for the partial differential equations which renders the treatment of the axi-symmetrical problem in Section 2.5 as correct as that of the two-dimensional problem.

5. CONCLUSIONS

(a) The distribution of boundary-layer thickness and wall shear stress over a two-dimensional or axi-symmetrical body in a laminar flow, with prescribed free-stream and normal wall velocity distributions, can be approximately calculated by solution of a non-linear ordinary first-order differential equation. (Equations (17) or (29).)

(b) The calculation involves references to standard graphs or tables (Figs. 2, 3 and 4), which are based on the "similar" solutions of the laminar boundary-layer equation.

(c) In many cases, good accuracy may be obtained by methods in which numerical integration is replaced by a single quadrature (e.g. equation (23)). Repeated quadrature is often a quick procedure for obtaining the solution to the problem (equations (23), (24), etc.).

(d) The "similar" boundary layers can be characterized as those appropriate to free-stream velocity distributions obeying the law: $du_G/dx = Cu_G^n$, where C and n are constants. The non-dimensional variables in terms of which their solutions are expressed can be chosen conveniently so as to involve du_G/dx but not x itself.

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